

# Existence of optimal controls for SPDE with locally monotone coefficients <sup>1</sup>

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## Abstract

The aim of this paper is to investigate the existence of optimal controls for systems described by stochastic partial differential equations (SPDEs) with locally monotone coefficients controlled by different external forces which are feedback controls. We apply our result to various types of SPDEs such as the stochastic 2-D Navier-Stokes equation, the stochastic nonlocal equation, stochastic linear equations and stochastic semilinear equations.

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# 1 Introduction

Let  $H$  be a real separable Hilbert space. Let  $V$  be a reflexive Banach space. Identify  $H$  with its dual  $H'$  and denote the dual of  $V$  by  $V'$ . Let

$$V \subset H \cong H' \subset V'$$

where the inclusions are assumed to be dense and compact. The triad  $(H, V, V')$  is known as a *Gelfand triple*. We will denote by  $\|\cdot\|_V$ ,  $\|\cdot\|$ ,  $\|\cdot\|_{V'}$  the norms in  $V$ ,  $H$ , and  $V'$  respectively. The inner product in  $H$  and the duality scalar product between  $V$  and  $V'$  will be denoted by  $(\cdot, \cdot)$  and  $\langle \cdot, \cdot \rangle$  respectively.

Let  $\{W_t\}_{t \geq 0}$  be a cylindrical Wiener process on a separable Hilbert space  $U$  w.r.t. a complete filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  and  $(L_2(U; H), \|\cdot\|_2)$  denotes the space of all Hilbert-Schmidt operators from  $U$  to  $H$ .

Let  $T > 0$  be some fixed time. Consider the following initial value problem involving a controlled SPDE differential of the form:

$$du(t) = (A(t, u(t), u(t)) + \Phi(t, u(t)))dt + \Xi(t, u(t))dW(t), \quad u(0) = u_0 \quad (1)$$

where  $A : [0, T] \times V \times V \times \Omega \rightarrow V'$ ,  $\Phi : [0, T] \times H \times \Omega \rightarrow H$  and  $\Xi : [0, T] \times V \times \Omega \rightarrow L_2(U; H)$  are progressively measurable,  $A$  satisfies a locally monotone condition (see condition **A2** below) and  $\Phi$  is a control.

In this paper we will study the existence of an optimal control which minimizes the cost function  $\mathcal{J}(\Phi)$  with  $\Phi$  belonging to  $\mathcal{U}$ , the set of controls associated with the controlled initial value problem (1).

The problem of the existence of an optimal control for SPDEs is an important question in optimal control theory and often resolved by assuming that the set of admissible controls is compact and by using the Main Theorem for Minimum Problems (see [13], Theorem 38.B ). In order to answer

this question, we use a weaker condition to the set of admissible controls which is weak sequentially compact and similarly with the Theorem 38.A of Zeidler [13], we assume that the functional cost is weak sequentially lower semicontinuous. The problem of the existence of an optimal control for SPDEs has been studied by several authors, for example, by Nagase [10], Buckdahn and Răşcanu [3], Gatarek and Sobczyk [5], Guisepina and Federica [6], and Al-Hussein [1] but the results of these papers cannot be applied in the study of the equation in (1) because they assume semilinearity or boundedness for the nonlinearities. The existence of optimal controls for the stochastic Navier - Stokes equation was studied in [2]. Following the ideas of [2] we obtain a generalization demonstrating the existence of optimal control to a wide class of SPDEs with locally monotone coefficients. Thus, the paper [2] is a particular case of the present work. Furthermore, our result (Theorem 2.3) may also be applied to demonstrate the existence of optimal control to other locally monotone SPDEs as we will see in Section 3, it is worth pointing out that the Example 3.3 demonstrating the existence of optimal control to stochastic nonlocal parabolic equation is new in the literature. In spite of assumptions within the Theorem 2.3, the examples that we get demonstrating the existence of optimal control show us that these assumptions are not very restrictive.

The article is organized in the following way: in Section 2, we present the basic spaces, the norms, properties and notations which we are going to work with in the subsequent sections. We formulate the control problem, which is the goal of this work and we prove the existence of an optimal control. The idea is to prove that a minimizing sequence has a subsequence which converges weakly (see Lemma 2.1 ). Then, we prove that weak convergence of the feedback controls implies strong convergence of a sub-

quence of the corresponding solutions (see Theorems 2.1 and 2.2). Finally, in Section 3 we provide examples where our result is applied to the stochastic 2-D Navier-Stokes equation, the stochastic nonlocal equation and the stochastic semilinear equation. To simplify notation, we use the letter  $\mathbb{T}$  for the interval  $[0, T]$ . Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space,  $(\mathcal{F}_t)_{t \in \mathbb{T}}$  a right-continuous filtration such that  $\mathcal{F}_0$  contains all  $\mathcal{F}$ -null sets and let  $\mathbb{E}(X)$  denote the mathematical expectation of the random variable  $X$ . We abbreviate “*almost surely*  $\omega \in \Omega$ .” to a.s.

Let  $B$  be a Banach space with norm  $\|\cdot\|_B$  and let  $\mathcal{B}(B)$  denote the Borel  $\sigma$ -algebra of  $B$ . The space  $L^2(\Omega \times \mathbb{T}; B)$  is the set of all  $\mathcal{F} \otimes \mathcal{B}(\mathbb{T})$ -measurable processes  $u : \Omega \times \mathbb{T} \rightarrow B$  which are  $\mathcal{F}_t$ -adapted and  $\mathbb{E}(\int_{\mathbb{T}} \|u\|_B^2 dt) < \infty$ . The constant  $c_{HV}$  is such that  $\|v\|^2 \leq c_{HV} \|v\|_V^2$  for all  $v \in V$ .

In order to get solutions to (1), we state the following conditions on the coefficients: Suppose there exist constants  $\alpha > 1$ ,  $\beta \geq 0$ ,  $\theta > 0$ ,  $K > 0$  and a positive adapted process  $f \in L^1([0, T] \times \Omega; \mathbb{R})$  such that the following conditions hold for all  $v, v_1, v_2 \in V$  and a.e.  $(t, \omega) \in \mathbb{T} \times \Omega$ .

**A1)** (Hemicontinuity) The map  $s \rightarrow \langle A(t, v_1 + sv_2, v_1 + sv_2), v \rangle + \langle \Phi(t, v_1 + sv_2), v \rangle$  is continuous on  $\mathbb{R}$ .

**A2)** (Local monotonicity)

$$2\langle A(t, v_1, v_1) - A(t, v_2, v_2), v_1 - v_2 \rangle + 2\langle \Phi(t, v_1) - \Phi(t, v_2), v_1 - v_2 \rangle + \|\Xi(t, v_1) - \Xi(t, v_2)\|_2^2 \leq (K + \rho(v_2))\|v_1 - v_2\|^2,$$

where  $\rho : V \rightarrow [0, +\infty)$  is a measurable function and locally bounded in  $V$ .

**A3)** (Coercivity)

$$2\langle A(t, v_1, v_1), v_1 \rangle + 2\langle \Phi(t, v_1), v_1 \rangle + \|\Xi(t, v_1)\|_2^2 + \leq -\theta\|v_1\|_V^2 + K\|v\|^2 + f(t).$$

**A4)** (Growth)

$$\|A(t, v_1, v_1)\|_{V'}^2 + \|\Phi(t, v_1)\|_{V'}^2 \leq (f(t) + K\|v_1\|_V^2)(1 + \|v_1\|^\beta).$$

In this work, we understand that the stochastic process  $u_\Phi$  is a solution to the problem in (1) in the following sense.

**Definition 1.1** *Let  $u_0$  be a random variable which does not depend on  $W(t)$ . The stochastic process  $(u_\Phi(t))_{t \in \mathbb{T}} \in L^2(\Omega \times \mathbb{T}; V)$ ,  $\mathcal{F}_t$ -adapted, with a.s. sample paths continuous in  $H$ , is a solution to (1) if it satisfies the equation:*

$$\begin{aligned} (u_\Phi(t), v) = & (u_0, v) + \int_0^t \langle A(u_\Phi(s), v) \rangle ds + \int_0^t (\Phi(s, u_\Phi(s)), v) ds + \\ & + \int_0^t (v, (\Xi(s, u_\Phi(s)) dW(s)) \end{aligned} \quad (2)$$

a.s. for all  $v \in V$  and  $t \in \mathbb{T}$ .

**Uniqueness** means indistinguishability.

We need the following existence of solutions theorem, the result is a particular case of Theorem 1.1 of [8], but sufficient for our objective.

**Theorem 1.1** *Let  $u_0 \in L^4(\Omega, V)$ . Suppose that (A1) - (A4) is satisfied and there is a constant  $C$  such that*

$$\begin{aligned} \|\Phi(t, v)\|_{V'}^2 + \|\Xi(t, v)\|_2^2 &\leq C(f(t) + \|v\|^2), \quad t \in \mathbb{T}, v \in V; \\ \rho(v) &\leq C(1 + \|v\|_V^2)(1 + \|v\|^\beta) \quad v \in V. \end{aligned} \quad (3)$$

*The problem (2) has a unique solution  $u_\Phi$  which has a.s. sample paths continuous in  $H$ .*

**Proof:** See Theorem 1.1 of [8] . □

## 2 Formulation of the control problem and main result

We consider the controlled SPDE (1) controlled by *continuous feedback controls* and we denote by  $\mathcal{U} := \{\Phi : \mathbb{T} \times L^2(D) \rightarrow L^2(D)\}$  the set of the admissible controls satisfying:

$$\|\Phi(0, 0)\|^2 \leq \eta \quad a.s. \quad (4)$$

and for all  $s, t \in \mathbb{T}$ ,  $x, y \in H$

$$\|\Phi(t, x) - \Phi(s, y)\|^2 \leq \lambda|t - s|^2 + \alpha\|x - y\|^2 \quad a.s. \quad (5)$$

where  $\eta, \lambda, \alpha$  are positive constants.

Furthermore, we will assume that the coefficients of (1) satisfy the following conditions, for all  $v, v_1, v_2 \in V$  and a.e.  $(t, \omega) \in \mathbb{T} \times \Omega$ :

**C1)** there is a constant  $L > 0$  such that

$$\|\Xi(t, v_1) - \Xi(t, v_2)\|_2^2 \leq L\|v_1 - v_2\|^2 \text{ and } \|\Xi(t, v_1)\|_2 = 0$$

**C2)** there are nonnegative constants  $K_1$  and  $J_1$  such that

$$\langle A(t, v, v_1), v_1 \rangle \leq -K_1\|v_1\|_V^2 + J_1\|v_1\|^2$$

**C3)** there is a positive constant  $\theta_1$  such that

$$\langle A(t, v, v_1) - A(t, v, v_2), v_1 - v_2 \rangle \leq -\theta_1 \|v_1 - v_2\|_V^2$$

**C4)** there are nonnegative constants  $c_1, c_2, c_3, c_4, c_5, c_6$  and  $c_7$  such that

$$\begin{aligned} & \langle A(t, v_1, v_2) - A(t, v_3, v_3), v_2 - v_3 \rangle \leq \\ & \leq c_1\|v_2 - v_1\|_V\|v_1 - v_3\|_V^{1/2} \|v_1 - v_3\|^{1/2}\|v_2 - v_3\|_V^{1/2}\|v_2 - v_3\|^{1/2} + \\ & + c_2\|v_1\|\|v_1\|_V\|v_1 - v_2\|_V \|v_1 - v_2\| + (c_7 + c_3\rho(v_1))\|v_2 - v_3\|^2 + \\ & + c_4\|v_2 - v_1\|_V^2 + c_5\rho(v_1)\|v_1 - v_2\|_V^2 - c_6\|v_2 - v_3\|_V^2. \end{aligned}$$

**C5)** there are nonnegative constants  $\theta_2, p_3, p_4$  and  $p_5$  such that

$$\|A(t, v, v_1)\|_{V^*}^2 \leq \theta_2 \|v_1\|_V^2 + p_3 \|v\|^2 \|v\|_V^2 + p_4 \|v_1\|^2 \|v_1\|_V^2 + p_5$$

**Remark 2.1** Under the conditions (4), (5) and **(C1)** the solution  $u_\Phi$  obtained in the Theorem 1.1 satisfies:

$$\mathbb{E}(\sup_{t \in \mathbf{T}} \|u_\Phi(t)\|^2) + \mathbb{E}(\int_0^T \|u_\Phi(s)\|_V^2 ds) \leq c \mathbb{E}(\|u_0\|^2) \quad (6)$$

and

$$\mathbb{E}(\sup_{t \in \mathbf{T}} \|u_\Phi(t)\|^4) + \mathbb{E}\left(\int_0^T \|u_\Phi(s)\|^2 ds\right)^2 \leq c \mathbb{E}(\|u_0\|^4) \quad (7)$$

where  $c = c(L, \eta, \lambda, \alpha, \theta, T)$  is a positive constant.

Let us now define the *cost functional*

$$\mathcal{J}(\Phi) := \mathbb{E}(\int_0^T (\mathcal{L}(s, u_\Phi(s)) + \mathcal{K}(\Phi(s, u_\Phi(s)))) ds) + \mathbb{E}(\mathcal{H}(u_\Phi(T))), \quad \Phi \in \mathcal{U} \quad (8)$$

whenever the integral in (8) exists and is finite, with  $\mathcal{L} : \mathbb{T} \times H_0^1(D) \rightarrow \mathbb{R}_+$ ,  $\mathcal{K} : L^2(D) \rightarrow \mathbb{R}_+$ , and  $\mathcal{H} : L^2(D) \rightarrow \mathbb{R}_+$ . It is required that the mappings  $\mathcal{K}$ ,  $\mathcal{H}$ , and  $u \in L^2(\mathbb{T}; H_0^1(D)) \mapsto \int_0^T \mathcal{L}(s, u(s)) ds$  are weak sequentially lower semicontinuous.

Our control problem is to minimize (8) over  $\mathcal{U}$ , we denote by  $(\mathcal{P})$  the problem of minimizing  $\mathcal{J}$  among the admissible controls. Any  $\Phi^* \in \mathcal{U}$  satisfying  $\mathcal{J}(\Phi^*) = \inf\{\mathcal{J}(\Phi) : \Phi \in \mathcal{U}\}$  is called an *optimal control*.

The following lemma proves that given a minimizing sequence for the problem  $(\mathcal{P})$  we can obtain a subsequence and a mapping  $\Phi \in \mathcal{U}$ , such that the subsequence converges weakly to  $\Phi$ .

**Lemma 2.1** Let  $\Phi_n$  be a minimizing sequence for problem  $(\mathcal{P})$ . There exists a subsequence  $n_k$  of  $n$  and a mapping  $\Phi \in \mathcal{U}$  such that for all  $t \in \mathbb{T}$ ,  $x, y \in H$ , we have

$$\lim_{k \rightarrow \infty} (\Phi_{n_k}(t, x), y) = (\Phi(t, x), y). \quad (9)$$

**Proof:** See Lemma 4.1 of [2].  $\square$

For simplicity the subsequence of  $\{\Phi_{n_k}\}_{k=1}^\infty$  obtained in the previous lemma will be relabeled as the same, for this sequence and  $\Phi$  as in the last lemma let us consider the equation

$$\begin{aligned} (\hat{u}_{\Phi_n}(t), v) &= (u_0, v) + \int_0^t \langle A(u_\Phi(s), \hat{u}_{\Phi_n}(s)), v \rangle ds + \\ &+ \int_0^t (\Phi_n(s, u_\Phi(s)), v) ds + \int_0^t (v, \Xi(s, u_\Phi(s)) dW(s)) \end{aligned} \quad (10)$$

a.s.,  $v \in V$ ,  $t \in \mathbb{T}$  and for  $n \in \mathbb{Z}^+$ . Since the coefficients in the equation (10) satisfied the condition **(C2)**, **(C3)**, **(A1)**, **(A3)** and **(A4)**, there is a unique process  $\hat{u}_{\Phi_n} \in L^2(\Omega \times \mathbb{T}; V)$  which is a solution of (10) with a.s. continuous trajectories in  $H$  (see Theorem 4.2.4, p. 75 of [11] or Theorem 3.6, p. 32 of [7]) satisfying:

$$\mathbb{E}(\sup_{t \in \mathbb{T}} \|\hat{u}_{\Phi_n}(t)\|^4) + \mathbb{E} \left( \int_0^T \|\hat{u}_{\Phi_n}(s)\|_1^2 ds \right)^2 \leq c (\mathbb{E}(\|u_0\|^4) + \mathbb{E}(\int_0^T \|u_\Phi(s)\|^4 ds)) \quad (11)$$

where  $c$  is a positive constant independent of  $n$ .

To obtain the estimates in (11) we use the Burkholder and Schwarz inequalities.

**Theorem 2.1** *The solution to (2) and (10) satisfies:*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left( \int_0^T \|(u_\Phi - \hat{u}_{\Phi_n})(s)\|_V^2 ds \right) = \lim_{n \rightarrow \infty} \mathbb{E}(\|(u_\Phi - \hat{u}_{\Phi_n})(T)\|^2) = 0.$$

**Proof:** Let us consider the equation

$$(z(t), v) = (u_0, v) + \int_0^t \langle A(u_\Phi(s), z(s)), v \rangle ds + \int_0^t (v, \Xi(s, u_\Phi(s)) dW(s)) \quad (12)$$

a.s.,  $v \in V$  and  $t \in \mathbb{T}$ . By a similar argument as in the case of equation (10), there exists a unique solution  $z \in L^2(\Omega \times \mathbb{T}; V)$  of (12), which has a.s. continuous trajectories in  $H$ . By using the Gronwall lemma, we get the



estimate

$$\mathbb{E}(\sup_{t \in \mathbb{T}} \|z(t)\|^2) + 2p\mathbb{E}(\int_0^T \|z(s)\|_V^2 ds) \leq k \left( \mathbb{E}(\|u_0\|^2) + \mathbb{E}(\int_0^T \|u_\Phi(s)\|^2 ds) \right).$$

Then, there exists  $k_2(\omega) > 0$  and a.s.,

$$\begin{aligned} \sup_{t \in \mathbb{T}} \|z(t)\|^2 &\leq k_2(\omega), \\ \int_0^T \|z(s)\|_V^2 ds &\leq k_2(\omega) \end{aligned} \quad (13)$$

and

$$\begin{aligned} \sup_{t \in \mathbb{T}} \|u_\Phi(t)\|^2 &\leq k_2(\omega), \\ \int_0^T \|u_\Phi(s)\|_V^2 ds &\leq k_2(\omega). \end{aligned} \quad (14)$$

Using stochastic integral properties and (7), we obtain that for all  $s, t \in \mathbb{T}$ ,  $t > s$ ,

$$\mathbb{E}(\|\int_s^t \Xi(r, u_\Phi(r)) dW(r)\|_{V'}^4) \leq c(t-s)^2 E(\|u_0\|^4)$$

As a result of the Kolmogorov continuity test, we get a random variable  $\tilde{H}$  such that

$$\|\int_s^t \Xi(r, u_\Phi(r)) dW(r)\|_{V'}^2 \leq \tilde{H}(\omega) |t-s|^{2\gamma} \quad (15)$$

a.s. with  $0 < \gamma < \frac{1}{4}$  and for every  $t, s \in \mathbb{T}$ .

Let  $\bar{\Omega} \subset \Omega$  with  $\mathbb{P}(\bar{\Omega}) = 1$  such that for  $\omega \in \bar{\Omega}$  the equations in (2) and (12) are satisfied and, for each  $n \in \mathbb{Z}^+$ , (10) is also satisfied and the inequalities in (13), (14) and (15) are satisfied.

From (10), (12), (14) and the properties of  $A$  (**C3**) and  $\Phi_n$ , it follows that for  $\omega \in \bar{\Omega}$ ,

$$\begin{aligned} \sup_{t \in \mathbb{T}} \|(\hat{u}_{\Phi_n} - z)(t)\|^2 + \theta_1 \int_0^T \|(\hat{u}_{\Phi_n} - z)(s)\|_V^2 ds &\leq \frac{c_{HV}^2 2T(\lambda T^2 + \eta)}{\theta_1} + \\ &+ \frac{c_{HY}^2 2\alpha}{\theta_1} \int_0^T \|u_{\Phi(s)}\|_V^2 ds \leq k(\omega), \end{aligned}$$

where  $k(\omega)$  is independent of  $n$ . Hence, for all  $n \in \mathbb{Z}^+$ , we obtain

$$\sup_{t \in \mathbb{T}} \|\hat{u}_{\Phi_n}(t)\|^2 + p \int_0^T \|\hat{u}_{\Phi_n}(s)\|_V^2 ds \leq k(\omega) \quad (16)$$

for  $\omega \in \bar{\Omega}$ , where  $k(\omega)$  is a positive constant independent of  $n$ .

For  $\omega \in \bar{\Omega}$ , we consider the sequence

$$F(\omega) := \{\hat{u}_{\Phi_n}(\omega, \cdot)\}_{n=1}^{\infty},$$

which is bounded because of (16).

From (10), we obtain

$$\begin{aligned} \|\hat{u}_{\Phi_n}(t) - \hat{u}_{\Phi_n}(s)\|_{V'}^2 &\leq \left\| \int_s^t \Xi(r, u_{\Phi}(r)) dW(r) \right\|_{V'}^2 + \\ &+ (t-s) \int_s^t (\|A(u_{\Phi}(r), \hat{u}_{\Phi_n}(r))\|_{V'}^2 + \|\Phi_n(r, u_{\Phi}(r))\|_{V'}^2) dr, \end{aligned}$$

for each  $t, s \in \mathbb{T}$ ,  $t > s$ . From this, (15), (16) and the properties of  $A$  (C5),  $\Phi_n$ , we get

$$\|\hat{u}_{\Phi_n}(t) - \hat{u}_{\Phi_n}(s)\|_{V'}^2 \leq k(\omega)(t-s) + \tilde{H}(\omega)(t-s)^{2\gamma}$$

for  $\gamma \in (0, \frac{1}{4})$  and where  $k(\omega) > 0$  is independent of  $n$ .

Consequently,  $F(\omega)$  is equi-continuous in  $C([0, T], V')$ . Now, using Dubinsky's Theorem, (see Theorem 4.1, p. 132 of [12]), it follows that  $F(\omega)$  is relatively compact in  $L^2(0, T; H)$ . Thus, there exists a subsequence  $n_k$  of  $n$  and  $\hat{u} \in L^2(0, T; H)$  such that

$$\lim_{k \rightarrow \infty} \int_0^T \|(\hat{u}_{\Phi_{n_k}} - \hat{u})(s)\|^2 ds = 0. \quad (17)$$

From (10), (2) and the properties of  $A$  (C3) we obtain

$$\begin{aligned} \|\hat{u}_{\Phi_{n_k}}(T) - u_{\Phi}(T)\|^2 &+ 2\theta_1 \int_0^T \|(\hat{u}_{\Phi_{n_k}} - u_{\Phi})(t)\|_V^2 dt \leq \\ &\leq \int_0^T \left( \Phi_{n_k}(t, u_{\Phi}(t)) - \Phi(t, u_{\Phi}(t)), (\hat{u}_{\Phi_{n_k}} - \hat{u})(t) \right) dt + \\ &+ \int_0^T \left( \Phi_{n_k}(t, u_{\Phi}(t)) - \Phi(t, u_{\Phi}(t)), (\hat{u} - u)(t) \right) dt. \end{aligned}$$

We use Lemma 2.1, (17) and the properties of  $\Phi_n$  and  $\Phi$  to obtain

$$\lim_{k \rightarrow \infty} \|(\hat{u}_{\Phi_{n_k}} - u_{\Phi})(T)\|^2 = \lim_{k \rightarrow \infty} \int_0^T \|(\hat{u}_{\Phi_{n_k}} - u_{\Phi})(t)\|_V^2 dt = 0.$$

Since every subsequence of  $(\hat{u}_{\Phi_n}(\omega, \cdot))$  has a subsequence which converges to the same limit  $u_{\Phi}(\omega, \cdot)$  in the space  $L^2(0, T; V)$ , it follows that the sequence  $(\hat{u}_{\Phi_n}(\omega, \cdot))$  converges to  $u_{\Phi}(\omega, \cdot)$ . Similarly, we can conclude that  $(\hat{u}_{\Phi_n}(\omega, T))$  converges to  $u_{\Phi}(\omega, T)$  in  $H$ .

From Remark (2.1) and (11), the processes  $(\hat{u}_{\Phi_n})_{t \in \mathbb{T}}$  and  $(u_{\Phi})_{t \in \mathbb{T}}$  are uniformly integrable and thus the theorem follows.  $\square$

Let  $(Q(t))$  be a  $H_0^1(D)$ -valued process with

$$\int_0^T \|Q(s)\|_1^2 ds < \infty \text{ and } \sup_{t \in \mathbb{T}} \|Q(t)\|^2 < \infty \text{ a.s.}$$

For each  $M$ , a nonnegative integer, we define the following stopping times:

$$\bar{\mathcal{T}}_M^Q := \begin{cases} \inf \left\{ t \in \mathbb{T} : \int_0^t \|Q(s)\|_V^2 ds \geq M \right\}, \\ T, \text{ if } \int_0^T \|Q(s)\|_V^2 ds < M, \end{cases}$$

and

$$\hat{\mathcal{T}}_M^Q := \begin{cases} \inf \{ t \in \mathbb{T} : \sup_{t \in \mathbb{T}} \|Q(t)\|^2 \geq M \} \\ T, \text{ if } \sup_{t \in \mathbb{T}} \|Q(t)\|^2 < M \end{cases}$$

and  $\mathcal{T}_M^Q := \min \{ \bar{\mathcal{T}}_M^Q, \hat{\mathcal{T}}_M^Q \}$ .

Let  $\Phi_n$  and  $\Phi$  be the sequence and the map obtained in the Lemma 2.1, the following theorem asserts that there is a subsequence  $n_k$  of  $n$  such that the correspondent solutions of (2)  $u_{\Phi_{n_k}}$  converge strongly to  $u_{\Phi}$ .

**Theorem 2.2** *Let  $\{\Phi_n\}_{n \in \mathbb{N}}$  be as in the last theorem. There is a subsequence  $n_k$  of  $n$  such that*

$$\lim_{k \rightarrow \infty} \mathbb{E} \left( \int_0^T \|(u_{\Phi} - u_{\Phi_{n_k}})(s)\|_1^2 ds \right) = \lim_{k \rightarrow \infty} \mathbb{E} (\|(u_{\Phi} - u_{\Phi_{n_k}})(T)\|^2) = 0.$$

**Proof:** For the sake of convenience, we use the abbreviations,  $u := u_{\Phi}$  and

$\mathcal{T}_M := \mathcal{T}_M^u$  for  $M = 1, 2, \dots$

Let  $e(t) := \exp \left( \int_0^t -2c_7 - \frac{2\alpha}{c_6} - 2L - 2c_3 \rho(u(s)) ds \right)$ . As a result of the Itô

formula, we get

$$\begin{aligned}
e(\mathcal{T}_M) \|\hat{u}_{\Phi_n}(\mathcal{T}_M) - u_{\Phi_n}(\mathcal{T}_M)\|^2 &= \int_0^{\mathcal{T}_M} e(s) \|\Xi(s, u(s)) - \Xi(s, u_{\Phi_n}(s))\|_2^2 ds + \\
&+ 2 \int_0^{\mathcal{T}_M} e(s) (\Phi_n(s, u(s)) - \Phi_n(s, u_{\Phi_n}(s)), (\hat{u}_{\Phi_n} - u_{\Phi_n})(s)) ds + \\
&+ 2 \int_0^{\mathcal{T}_M} e(s) \langle A(u(s), \hat{u}_{\Phi_n}(s)) - A(u_{\Phi_n}(s), u_{\Phi_n}(s)), (\hat{u}_{\Phi_n} - u_{\Phi_n})(s) \rangle ds + \\
&+ 2 \int_0^{\mathcal{T}_M} e(s) ((\hat{u}_{\Phi_n} - u_{\Phi_n})(s), \Xi(s, u(s)) - \Xi(s, u_{\Phi_n}(s))) dW(s) + \\
&+ \int_0^{\mathcal{T}_M} e'(s) \|(\hat{u}_{\Phi_n} - u_{\Phi_n})(s)\|^2 ds.
\end{aligned}$$

Using the properties of  $A$ , **(C4)**,  $\Phi_n$  and  $\Xi$ , we get

$$\begin{aligned}
&\mathbb{E}(e(\mathcal{T}_M) \|\hat{u}_{\Phi_n}(\mathcal{T}_M) - u_{\Phi_n}(\mathcal{T}_M)\|^2) + c_6 \mathbb{E}(\int_0^{\mathcal{T}_M} e(s) \|(\hat{u}_{\Phi_n} - u_{\Phi_n})(s)\|_V^2 ds) \leq \\
\leq &\mathbb{E}(\int_0^{\mathcal{T}_M} e'(s) \|(\hat{u}_{\Phi_n} - u_{\Phi_n})(s)\|^2 ds) + 2c_3 \mathbb{E}(\int_0^T e(s) \rho(u(s)) \|(\hat{u}_{\Phi_n} - u_{\Phi_n})(s)\|^2 ds) + \\
&+ 2c_4 \mathbb{E}(\int_0^T e(s) \|(\hat{u}_{\Phi_n} - u)(s)\|_V^2 ds) + 2c_5 \mathbb{E}(\int_0^T e(s) \rho(u(s)) \|(\hat{u}_{\Phi_n} - u)(s)\|_V^2 ds) + \\
&+ 2c_1 (\mathbb{E}(\int_0^T \|(\hat{u}_{\Phi_n} - u_{\Phi_n})(s)\|_V^2 ds) \mathbb{E}(\int_0^T \|(\hat{u}_{\Phi_n} - u_{\Phi_n})(s)\|^2 ds) + \mathbb{E}(\int_0^T \|(\hat{u}_{\Phi_n} - u)(s)\|_V^2 ds) + \\
&+ \mathbb{E}(\int_0^T \|(\hat{u}_{\Phi_n} - u)(s)\|^2 ds))^{1/2} (\mathbb{E}(\int_0^T \|(\hat{u}_{\Phi_n} - u)(s)\|_V^2 ds))^{1/2} + \\
&+ c_2 M (\mathbb{E}(\int_0^T \|(\hat{u}_{\Phi_n} - u)(s)\|_V^2 ds))^{1/2} (\mathbb{E}(\sup_{t \in \mathbb{T}} \|(\hat{u}_{\Phi_n} - u)(s)\|^2))^{1/2} + \\
&+ \frac{\alpha}{c_6} \mathbb{E}(\int_0^T e(s) \|(u - u_{\Phi_n})(s)\|^2 ds) + L \mathbb{E}(\int_0^{\mathcal{T}_M} e(s) \|(u - u_{\Phi_n})(s)\|^2 ds) + \\
&+ 2c_7 \mathbb{E}(\int_0^T e(s) \|(\hat{u}_{\Phi_n} - u_{\Phi_n})(s)\|^2 ds) .
\end{aligned} \tag{18}$$

From Theorem 2.1, we can get a subsequence  $\{\hat{u}_{\Phi_{n_k}}\}_{k=1}^{\infty}$  that converges to  $u$  a.e.  $(\omega, t) \in \Omega \times \mathbf{T}$ . Thus, from (18), we obtain

$$\begin{aligned}
E((e(\mathcal{T}_M)\|\hat{u}_{\Phi_{n_k}}(\mathcal{T}_M) - u_{\Phi_{n_k}}(\mathcal{T}_M)\|^2) + c_6 E(\int_0^{\mathcal{T}_M} e(s)\|(\hat{u}_{\Phi_{n_k}} - u_{\Phi_{n_k}})(s)\|_V^2 ds) \leq \\
\leq 2c_5 E(\int_0^{\mathcal{T}} e(s)\rho(u(s))\|(\hat{u}_{\Phi_{n_k}} - u)(s)\|_V^2 ds) + \\
+ 2c_1 \left( E(\int_0^{\mathcal{T}} \|(\hat{u}_{\Phi_{n_k}} - u)(s)\|_V^2 ds) \right)^{1/2} \\
\times \left( E(\int_0^{\mathcal{T}} \|(u_{\Phi_{n_k}} - u)(s)\|_V^2 ds + \int_0^{\mathcal{T}} \|(u_{\Phi_{n_k}} - u)(s)\|^2 ds + \right. \\
\left. + E(\int_0^{\mathcal{T}} \|(\hat{u}_{\Phi_{n_k}} - u_{\Phi_{n_k}})(s)\|_V^2 \|(\hat{u}_{\Phi_{n_k}} - u_{\Phi_{n_k}})(s)\|^2 ds) \right)^{1/2} + \\
+ c_2 M \left( E(\int_0^{\mathcal{T}} \|(\hat{u}_{\Phi_{n_k}} - u)(s)\|_V^2 ds) \right)^{1/2} \left( E(\sup_{t \in \mathbf{T}} \|(\hat{u}_{\Phi_{n_k}} - u)(s)\|^2) \right)^{1/2} + \\
+ (2c_4 + 2L + \frac{2\alpha}{c_6}) E(\int_0^{\mathcal{T}} e(s)\|(\hat{u}_{\Phi_{n_k}} - u)(s)\|_V^2 ds).
\end{aligned}$$

From this, Theorems 2.1 and the triangle inequality, we obtain

$$\lim_{k \rightarrow \infty} E(\int_0^{\mathcal{T}_M} \|(u_{\Phi} - u_{\Phi_{n_k}})(s)\|_1^2 ds) = \lim_{k \rightarrow \infty} E(\|(u_{\Phi} - u_{\Phi_{n_k}})(\mathcal{T}_M)\|^2) = 0,$$

which implies the desired conclusion.  $\square$

Finally, we are in a position to formulate our main result.

**Theorem 2.3** *Under the assumptions of Theorem 1.1, if, moreover, the conditions (C1)-(C2)) are satisfied, then there exists an optimal control for the problem  $(\mathcal{P})$ .*

**Proof:** Let  $\{\Phi_n\}$  be a minimizing sequence for the problem  $(\mathcal{P})$ . We apply Lemma (2.1) and Theorem (2.2) to this sequence. Thus, there exists a subsequence  $\{\Phi_{n_k}\}$  of  $\{\Phi_n\}$  and  $\Phi \in \mathcal{U}$  such that, for all  $t \in \mathbf{T}$ ,  $x, y \in L^2(D)$  and a.s.  $\omega \in \Omega$ , the following hold:

$$\lim_{k \rightarrow \infty} (\Phi_{n_k}(t, u_{\Phi_{n_k}}), y) = (\Phi(t, u_{\Phi}), y)$$

and

$$\lim_{k \rightarrow \infty} \int_0^{\mathcal{T}} \|(u_{\Phi_{n_k}} - u_{\Phi})(s)\|_V^2 ds = \lim_{k \rightarrow \infty} \|(u_{\Phi_{n_k}} - u_{\Phi})(\mathcal{T})\|^2 = 0.$$

From Theorem (2.2) and the weak sequentially lower semicontinuous properties of  $\mathcal{L}$ ,  $\mathcal{K}$  and  $\mathcal{H}$ , we get

$$\mathcal{J}(\Phi) \leq \liminf_{k \rightarrow \infty} \mathcal{J}(\Phi_{n_k}).$$

Since  $\{\Phi_n\}$  is a minimizing sequence for the problem  $(\mathcal{P})$ ,  $\mathcal{J}(\Phi) = \min_{\lambda \in \mathcal{U}} \mathcal{J}(\lambda)$  and thus  $\Phi \in \mathcal{U}$  is an optimal feedback control for problem  $(\mathcal{P})$ .  $\square$

### 3 Examples

**Example 3.1** *The main result can be applied to the initial value problem involving the linear stochastic evolution equation:*

$$du(t) = (\mathcal{A}(t, u(t)) + \Phi(t, u(t)))dt + \Xi(t, u(t))dW(t), \quad u(0) = u_0 \quad (19)$$

where  $\mathcal{A} : \mathbb{T} \times V \times \Omega \rightarrow V'$  is a linear operator. Furthermore, we will suppose that there are constants  $\alpha_1$ ,  $\beta_1$  and  $\gamma_1$  such that a.e.  $(t, \omega) \in \mathbb{T} \times \Omega$  and  $v_1, v_2 \in V$ :

$$1) \quad |\langle \mathcal{A}(t, v_1), v_2 \rangle| \leq \alpha_1 \|v_1\|_V \|v_2\|_V$$

$$2) \quad |\langle \mathcal{A}(t, v_1), v_1 \rangle| \leq -\beta_1 \|v_1\|_V^2 + \gamma_1 \|v_1\|^2. \text{ Then, there is an optimal control } \Phi \text{ which minimizes the cost functional } \mathcal{J} \text{ given by (8).}$$

**Proof:** Under the conditions (1), (2) (above), (4), (5) and **(C1)** it is not hard to prove that the coefficients of the equation (19) satisfy the conditions **(A1)**, **(A2)** with  $K = 2\gamma_1 + \frac{2L}{\theta_1} + \alpha$  and  $\rho(v_2) = 0$  and **(A3)** with  $\theta = \beta_1$ ,  $K = \gamma_1$   $f \equiv 1$ , and **(A4)** with  $\beta = 2$ . Thus from the Theorem 1.1 there is a solution  $u_\Phi$  to the equation (19).

Taking  $A(t, u, v) = \mathcal{A}(v)$  we have that the coefficients of the equation (19) satisfy **(C2)** with  $K_1 = \beta_1$ ,  $J_1 = 0$ , **(C3)** with  $\theta_1 = \beta_1$ , **(C4)** with  $c_1 =$

$c_2 = c_3 = c_4 = 0 = c_5 = 0$ ,  $c_6 = \beta_1$ ,  $c_7 = \gamma_1$ , and **(C5)** with  $\theta_2 = \alpha_1$ ,  $p_3 = p_4 = p_5 = 0$ . So that the claim follows from Theorem 2.3.  $\square$

Let  $\mathcal{O}$  be a bounded domain in  $\mathbb{R}^2$  with smooth boundary. Define

$$V = \{v \in H_0^1(\mathcal{O}, \mathbb{R}^2) : \nabla \cdot v = 0 \text{ a.e. in } \mathcal{O}\},$$

with the norm

$$\|v\|_V := \left( \int_{\mathcal{O}} |\nabla v|^2 dx \right)^{1/2}$$

and  $H$  is the closure of  $V$  in the following norm

$$\|v\|_H := \left( \int_{\mathcal{O}} |v|^2 dx \right)^{1/2}.$$

The linear operator  $P_H$  (Helmholtz-Hodge projection) and  $\mathcal{A}$  (Stokes operator with viscosity constant  $\nu$ ) are defined by

$$\begin{aligned} P_H &: L^2(\mathcal{O}, \mathbb{R}^2) \rightarrow H \text{ orthogonal projection;} \\ \mathcal{A} &: H^2(\mathcal{O}, \mathbb{R}^2) \cup V \rightarrow H, \quad \mathcal{A}u = \nu P_H \Delta u. \end{aligned}$$

and the nonlinear operator

$$B : \mathcal{D}_B \subset H \times V \rightarrow H, \quad B(u, v) = P_H(u \cdot \nabla v) \quad (20)$$

with notation  $B(u) = B(u, u)$ .

It is well known then that the Navier-Stokes can be reformulated in the following abstract form

$$\frac{\partial u}{\partial t} = \mathcal{A}u + B(u) + f, \quad \text{in } L^2(0, T; V'), \quad (21)$$

with the initial condition

$$u(0) = u_0 \in H, \quad (22)$$

where  $f \in L^2(0, T; V')$  denotes some external force. It is standard that for the Gelfand triple

$$V \subset H = H' \subset V',$$

the following maps

$$\mathcal{A} : V \rightarrow V', \quad B : V \times V \rightarrow V'$$

are well defined and satisfied

$$\langle B(u, v), w \rangle = -\langle B(u, w), v \rangle, \quad \langle B(u, v), v \rangle = 0 \quad u, v, w \in V.$$

Now, we consider the following initial value problem stochastic 2-D Navier-Stokes equation

$$du(t) = (\mathcal{A}u(t) + B(u(t)) + \Phi(t, u(t)))dt + \Xi(t, X(t))dW(t), \quad u(0) = u_0 \quad (23)$$

where  $W$  is a Wiener process in  $H$  and  $\Phi \in \mathcal{U}$  is a control.

**Example 3.2** (*Stochastic 2-D Navier-Stokes equation*) *There is an optimal control  $\Phi$  which minimizes the cost functional  $\mathcal{J}$  given by 8 (see [2]).*

**Proof:** In this example we will consider  $A(t, u, v) = \mathcal{A}v + B(u, v)$  for  $t \in \mathbb{T}$ ,  $u, v \in V$ . First, we will verify that if  $u_0 \in L^4(\Omega, H)$ , then (23) has a unique solution  $u = u_\Phi$ . In fact, the hemicontinuity **(A1)** is due to the fact that  $B$  is a bilinear map.

About **(A2)**, we have that (see [9])

$$|\langle B(u, v), w \rangle| \leq 2\|w\|_V \|u\|^{1/2} \|u\|_V^{1/2} \|v\|^{1/2} \|v\|_V^{1/2} \quad \text{for all } u, v, w \in V \quad (24)$$

then

$$\begin{aligned} \langle B(u), u - v \rangle &= \langle B(u, v), u - v \rangle = -\langle B(u - v, u - v), v \rangle + \langle B(v), u - v \rangle \leq \\ &\leq \frac{4}{\nu} \|v\|_V^2 \|v - u\|^2 + \frac{\nu}{4} \|v - u\|_V^2 + \langle B(v), u - v \rangle \end{aligned}$$

then we obtain the following inequality (see Lemma 2.3 of [9], for a similar inequality)

$$\langle B(u) - B(v), u - v \rangle \leq \frac{4}{\nu} \|v\|_V^2 \|v - u\|^2 + \frac{\nu}{4} \|v - u\|_V^2 \quad (25)$$



thus

$$2\langle \mathcal{A}u - \mathcal{A}v, u - v \rangle + 2\langle B(u) - B(v), u - v \rangle + 2\langle \Phi(t, u) - \Phi(t, v), u - v \rangle + \\ + \|\Xi(t, u) - \Xi(t, v)\|_2^2 \leq -\nu\|u - v\|_V^2 + \frac{8}{\nu}\|v\|_V^2\|v - u\|^2 + L\|v - u\|^2 + \frac{4\alpha}{\nu}\|v - u\|^2$$

Hence, we have the local monotonicity **(A2)** with  $\rho(v) = \frac{2}{\nu}\|v\|_V^2$ .

As in ([8]) we can use the inequalities

$$\|u\|_{L^4}^2 \leq \sqrt{2}\|u\|_{L^2}\|\nabla u\|_{L^2} \quad u \in H_0^1(\mathcal{O}) \quad (26)$$

and (see Lemma 2.1 and proof of the Lemma 2.2 of [9])

$$|\langle B(u), w \rangle| \leq \sqrt{2}\|u\|_{L^4(\mathcal{O}; \mathbb{R}^2)}\|w\|_V, \text{ for } \|w\|_V \leq 1$$

to get **(A4)** with  $\beta = 2$ . Analogously **(A3)** is verified using the property  $\langle B(u, v), v \rangle = 0$ . Thus, from Theorem (1.1) there is a unique solution for the equation (23).

Now, we proceed to demonstrate **(C2)** - **(C5)**. In fact, since

$$\langle B(v_1, v_2) - B(v_3, v_3), v_2 - v_3 \rangle = \langle B(v_1 - v_3, v_2 - v_1), v_2 - v_3 \rangle + \\ + \langle B(v_1 - v_2, v_1), v_2 - v_3 \rangle + \langle B(v_2 - v_3, v_1), v_2 - v_3 \rangle \leq \\ \leq 2\|v_2 - v_1\|_V \|v_1 - v_3\|_V^{1/2}\|v_1 - v_3\|^{1/2}\|v_2 - v_3\|_V^{1/2}\|v_2 - v_3\|^{1/2} + \\ + \frac{4}{\nu}\|v_1\|\|v_1\|_V \|v_1 - v_2\|_V\|v_1 - v_2\| + \frac{4}{\nu}\|v_1\|_V^2\|v_2 - v_3\|^2 + \frac{\nu}{2}\|v_2 - v_3\|_V^2,$$

thus we have

$$2\langle \mathcal{A}v_2 - \mathcal{A}v_3, v_2 - v_3 \rangle + 2\langle B(v_1, v_2) - B(v_3, v_3), v_2 - v_3 \rangle \leq \\ \leq 4\|v_2 - v_1\|_V\|v_1 - v_3\|_V^{1/2}\|v_1 - v_3\|^{1/2}\|v_2 - v_3\|_V^{1/2}\|v_2 - v_3\|^{1/2} + \\ + \frac{8}{\nu}\|v_1\|\|v_1\|_V\|v_1 - v_2\|_V\|v_1 - v_2\| + \frac{8}{\nu}\|v_1\|_V^2\|v_2 - v_3\|^2 - \nu\|v_2 - v_3\|_V^2,$$

thus **(C4)** is satisfied with  $c_1 = 4$ ,  $c_2 = \frac{8}{\nu}$ ,  $c_3 = 4$ ,  $c_4 = c_5 = c_7 = 0$  and  $c_6 = \nu$ . Since  $\langle B(v, v_1), v_1 \rangle = 0$ , we have that **(C2)** is satisfied with  $K_1 = \nu$  and  $J_1 = 0$ . To demonstrate **(C3)**, observe that

$$\langle B(v, v_1) - B(v, v_2), v_1 - v_2 \rangle = \langle B(v, v_1 - v_2), v_1 - v_2 \rangle = 0,$$

hence

$$\langle \mathcal{A}v_1 - \mathcal{A}v_2, v_1 - v_2 \rangle + \langle B(v, v_1) - B(v, v_2), v_1 - v_2 \rangle \leq -\nu \|v_1 - v_2\|_V^2,$$

thus, **(C3)** is satisfied with  $\theta_1 = \nu$ . In view of (24) we have, for  $v \in V$  fixed

$$\|A(t, v, v_1)\|_V^{2'} \leq 2\nu \|v_1\|_V^2 + 4\|v\|_V^2 \|v\|^2 + 4\|v_1\|_V^2 \|v_1\|^2 \text{ for } v_1 \in V$$

hence **(C5)** is satisfied with  $\theta_2 = 2\nu$ ,  $p_3 = p_4 = 4$  and  $p_5 = 0$ . And the claim follows from Theorem 2.3.  $\square$

Now we will consider the following initial-boundary value problem

$$\begin{cases} du(t) = (a(\int_D u dx) \Delta u + \Phi(t, u)) dt + g(t, u) dW(t) & \text{on } t \in ]0, T[, \\ u(x, 0) = u_0(x) & \text{on } D \text{ and } u(x, t) = 0 & \text{on } \partial D \times ]0, T[ \end{cases} \quad (27)$$

where  $D$  is a bounded open subset of  $\mathbb{R}^n$  with smooth boundary  $\partial D$ ,  $n \geq 1$ ,  $a = a(s)$  is a continuous function with Lipschitz constant  $L$  such that  $0 < p \leq a(s) \leq P$  where  $p$  and  $P$  are constants,  $W$  is a Wiener process in  $L^2(D)$  and  $\Phi \in \mathcal{U}$  is a control.

In this case the Gelfand triple

$$V \subset H = H' \subset V',$$

where  $V = H_0^1(D)$  and  $H = L^2(D)$ .

**Example 3.3** (*Stochastic nonlocal parabolic equation*) *There is an optimal control  $\Phi$  which minimizes the cost functional  $\mathcal{J}$  given by (8) to the equation (27).*

**Proof:** In this example we will consider  $A(t, u, v) = (a(\int_D u dx) \Delta v$  for  $t \in \mathbb{T}$ ,  $u, v \in V$ . First, we will verify that if  $u_0 \in L^4(\Omega, H)$  then (27) has a unique solution  $u = u_\Phi$ . In fact, the hemicontinuity **(A1)** is a consequence of properties of  $a$ .

About **(A2)**, we have

$$\begin{aligned} \langle A(t, u, u) - A(t, v, v), u - v \rangle &\leq -\langle a(\int_D u(x)dx) \nabla u - a(\int_D v(x)dx) \nabla v, \nabla(u - v) \rangle \leq \\ &\leq -\langle a(\int_D u(t, x)dx) (\nabla u - \nabla v), \nabla(u - v) \rangle + \\ &\quad -\langle (a(\int_D u(t, x)dx) - a(\int_D v(t, x)dx)) \nabla v, \nabla(u - v) \rangle, \end{aligned}$$

then

$$\langle A(t, u, u) - A(t, v, v), u - v \rangle + \frac{p}{2} \|\nabla(u - v)\|^2 \leq \frac{C(D)L_1}{2p} \|u - v\|^2 \|v\|_V^2,$$

thus

$$\begin{aligned} 2\langle A(t, u) - A(t, v), u - v \rangle + 2\langle \Phi(t, u) - \Phi(t, v), u - v \rangle + \|\Xi(t, u) - \Xi(t, v)\|_2^2 \leq \\ \leq \frac{C(D)L_1}{p} \|v\|_V^2 \|v - u\|^2 + L\|v - u\|^2 + \frac{2\alpha}{p} \|v - u\|^2. \end{aligned}$$

Hence, we have the local monotonicity **(A2)** with  $\rho(v) = \frac{C(D)L_1}{p} \|v\|_V^2$  where  $C(D) = 1_D$ .

We proceed to demonstrate **(A4)**, we have

$$|\langle A(t, u), w \rangle|^2 \leq P \|u\|_V^2 \text{ for } \|w\|_V \leq 1$$

so we have **(A4)** with  $\beta = 2$ . Similarly, **(A3)** is verified. Thus, from Theorem (1.1) there is a unique solution for the equation (27).

The properties of  $a$  provide **(C2)** with  $K_1 = p$  and  $J_1 = 0$ . Using the properties of  $a$  we obtain **(C3)** with  $\theta_1 = p$ . Now, we proceed to demonstrate **(C4)**. In fact, since

$$\begin{aligned} \langle A(v_1, v_2) - A(v_3, v_3), (v_2 - v_3) \rangle &= \\ &= \langle A(v_1, v_2) - A(v_3, v_2), (v_2 - v_3) \rangle + \\ &\quad + \langle A(v_3, v_2 - v_3), (v_2 - v_3) \rangle = \\ &= \left( (a(\int_D v_1(x)dx) - a(\int_D v_3(x)dx)) \nabla v_2 - \nabla v_1, \nabla(v_2 - v_3) \right) + \\ &\quad - \left( (a(\int_D v_1(x)dx) - a(\int_D v_3(x)dx)) \nabla v_1, \nabla(v_2 - v_3) \right) + \\ &\quad + \langle A(v_3, v_2 - v_3), (v_2 - v_3) \rangle \leq \\ &\leq -\frac{3p}{4} \|v_2 - v_3\|_V^2 + \frac{2P}{p} \|v_2 - v_1\|_V^2 + \\ &\quad + \frac{4L_1^2 C(D)}{p} \|v_1\|_V^2 \|v_2 - v_3\|^2 + \frac{4L_1^2 C(D)}{p} \|v_1\|_V^2 \|v_1 - v_2\|_V^2 \end{aligned}$$

thus (C4) is satisfied with  $c_1 = 0$ ,  $c_2 = 0$ ,  $c_3 = 4L_1$ ,  $c_4 = \frac{2P}{p}$ ,  $c_5 = 4L_1$ ,  $c_6 = \frac{3p}{4}$  and  $c_7 = 0$ . Using the properties of  $a$  we obtain (C5) with  $\theta_2 = P$  and  $p_3 = p_4 = p_5 = 0$ , and the claim follows from Theorem 2.3.  $\square$

Let  $D \subset \mathbb{R}^n$  be an open bounded domain with smooth boundary.

**Lemma 3.1** *Consider the Gelfand triple*

$$V := H_0^1(D) \subset H := L^2(D) \subset V' := H^{-1}(D)$$

and the operator

$$A(u) = \Delta u + \sum_{i=1}^d f_i(u) D_i u,$$

where  $f_i$ , for  $i = 1, \dots, d$  are bounded Lipschitz functions on  $\mathbb{R}$ .

(1) If  $d < 3$ , there exists a constant  $K_2$  such that

$$2\langle A(u) - A(v), u - v \rangle \leq -\|u - v\|_V^2 + (K_2 + K_2\|v\|_V^2)\|u - v\|_H^2, \quad u, v \in V.$$

(2) If  $d = 3$ , there exists a constant  $K_3$  such that

$$2\langle A(u) - A(v), u - v \rangle \leq -\|u - v\|_V^2 + (K_3 + K_3\|v\|_V^4)\|u - v\|_H^2, \quad u, v \in V.$$

(3) If  $f_i$  are independent of  $u$  for  $i = 1, \dots, d$ , i.e.

$$A(u) = \Delta u + \sum_{i=1}^d f_i D_i u,$$

then for  $d \geq 1$  we have

$$2\langle A(u) - A(v), u - v \rangle \leq -\|u - v\|_V^2 + K_4\|u - v\|_H^2, \quad u, v \in V.$$

where  $K_4$  is a constant.

**Proof:** See Lemma 3.1 of [8]  $\square$

**Example 3.4** (*Stochastic semi-linear equations*). Let  $d \leq 3$  and consider the initial value problem involving the controlled semi-linear stochastic equation

$$du(t) = (\Delta u(t) + \sum_{i=1}^d f_i(u(t)) D_i u(t) + \Phi(u(t))) dt + \Xi(u(t)) dW(t), \quad u(0) = u_0 \quad (28)$$

where  $W(t)$  is a Wiener process on  $L^2(D)$ ,  $\Phi$  is the control and  $f_i$ , are bounded Lipschitz functions on  $\mathbb{R}$  for  $i = 1, \dots, d$ . Suppose that  $|f_i(x)| \leq J < 1$ . There is an optimal control  $\Phi$  for the problem  $(\mathcal{P})$ .

**Proof:** We can suppose that all  $f_i$  with  $i = 1, \dots, d$  have a Lipschitz constant  $L_1$ .

We define the map

$$A(u, v) = \Delta v + \sum_{i=1}^d f_i(u(t)) D_i v(t), \quad u \in V.$$

The hemicontinuity **(A1)** follows from the continuity of  $f$  and  $\Xi$ . We give the proof of **(A2)**-**(A4)** only for the case  $d = 3$ ; the case  $1 \leq d < 3$  is similar. Therefore, by Lemma 3.1

$$2\langle A(u) - A(v), u - v \rangle + 2\langle \Phi(u) - \Phi(v), u - v \rangle \leq -\frac{1}{2} \|u - v\|_V^2 + (\alpha 2 + K_2 \|v\|_V^4) \|u - v\|_H^2$$

for  $u, v \in V$ . Hence, **(A2)** and **(A3)** are satisfied with  $\alpha = 2$  and  $\rho(v) = K_2 \|v\|_V^4$  respectively. We proceed to demonstrate **(A4)**, we have that

$$|\langle Au, v \rangle|^2 \leq C \|u\|_V^2 \|v\|_V^2$$

so we have **(A4)** with  $\beta = 2$ . Thus, from Theorem (1.1) there is a unique solution for the equation (28).

Now we proceed to verify **(C2)** - **(C5)**. Using the properties of  $f_i$  we have that **(C2)** is satisfied with  $K_1 = 1 - J$  and  $J_1 = 0$ . Since

$$\langle A(v, v_1) - A(v, v_2), v_1 - v_2 \rangle \leq -(1 - J) \|v_1 - v_2\|_V^2$$

we have that **(C3)** is satisfied with  $\theta_1 = 1 - J$ .

Using the properties of  $f_i$  we obtain the following inequality

$$\begin{aligned}
\langle A(v_1, v_2) - A(v_3, v_3), v_2 - v_3 \rangle &\leq -\|v_2 - v_3\|_V^2 + \\
\sum_{i=1}^d \int_D (f_i(v_1) - f_i(v_3))(D_i v_2 - D_i v_1)(v_2 - v_3) dx &+ \\
+ \sum_{i=1}^d \int_D (f_i(v_1) - f_i(v_2)) D_i(v_1)(v_2 - v_3) dx &+ \\
+ \sum_{i=1}^d \int_D (f_i(v_2) - f_i(v_3)) D_i(v_1)(v_2 - v_3) dx &+ \\
+ \sum_{i=1}^d \int_D f_i(v_3)(D_i v_2 - D_i v_3)(v_2 - v_3) dx &\leq \\
\leq -\|v_2 - v_3\|_V^2 + 2J\|v_2 - v_3\| \|v_2 - v_1\|_V &+ \\
+ L_1\|v_1 - v_2\|_{L^4(D)} \|v_2 - v_3\|_{L^4(D)} \|v_1\|_V &+ \\
+ L_1\|v_2 - v_3\|_{L^4(D)}^2 \|v_1\|_V + J\|v_2 - v_3\| \|v_2 - v_3\|_V, &
\end{aligned} \tag{29}$$

for  $v_1, v_2, v_3 \in V$ . For  $d < 3$ , from inequality (26) and from (29) we have

$$\begin{aligned}
\langle A(v_1, v_2) - A(v_3, v_3), v_2 - v_3 \rangle &\leq -\frac{1}{4}\|v_2 - v_3\|_V^2 + \\
+ \|v_2 - v_3\|^2 \|v_1\|^2 + 2J^2\|v_2 - v_3\|^2 &+ (1 + \frac{L_1^4}{2})\|v_2 - v_1\|_V^2 + \\
+ \frac{1}{2}\|v_1 - v_2\|_V^2 \|v_1\|_V^2 &+
\end{aligned}$$

thus **(C4)** is satisfied with  $c_1 = c_2 = 0$ ,  $c_3 = \frac{1}{K_2}$ ,  $c_4 = (1 + \frac{L_1^4}{2})$ ,  $c_5 = \frac{1}{2K_2}$ ,  $c_6 = \frac{1}{4}$  and  $c_7 = 2J^2$ .

For  $d = 3$ , from (29), Young's inequality and the following inequality (see p. 34 of [9]):

$$\|u\|_{L^4(D)}^4 \leq 4 \leq \|u\|_{L^2(D)} \|\nabla u\|_{L^2(D)}^3 \quad u \in V$$

we get

$$\begin{aligned}
\langle A(v_1, v_2) - A(v_3, v_3), v_2 - v_3 \rangle &\leq -\frac{1}{4}\|v_2 - v_3\|_V^2 + \\
+ L_1^4 3^3 (\frac{1+2^8}{2^6}) \|v_2 - v_3\|^2 \|v_1\|^4 &+ 2J^2\|v_2 - v_3\|^2 + (1 + \frac{3L_1^4}{4})\|v_2 - v_1\|_V^2 + \\
+ \frac{1}{4}\|v_1 - v_2\|_V^2 \|v_1\|_V^4 &+
\end{aligned}$$

thus **(C4)** is satisfied with  $c_1 = 0$ ,  $c_2 = 0$ ,  $c_3 = \frac{1}{K_3} L_1^4 3^3 (\frac{1+2^8}{2^6})$ ,  $c_4 = 1 + \frac{3L_1^4}{4}$ ,  $c_5 = \frac{1}{4K_3}$ ,  $c_6 = \frac{1}{4}$  and  $c_7 = 2J^2$ .

Finally, **(C5)** is satisfied with  $\theta_2 = 1 + L$ ,  $p_3 = p_4 = 0$  and  $p_5 = 1$ , and the claim follows from Theorem 2.3.  $\square$

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