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The evolution of cooperation in mixed games

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ABSTRACT

Cooperation has been studied in the context of game evolutionary theory by assuming that individuals play always the same game. Here we consider a mixture of two games G_1 and G_2. In each interaction of two individuals, they can play the games G_1 or G_2 with probabilities w and $1-w$, respectively. We define the evolutionary model and study the cooperation evolution in a well-mixed population and in a cycle. We show that in the well-mixed population the evolution is equivalent to the evolution given by the average game. In a cycle, we show that the intensity of selection plays an important role in the promotion or inhibition of cooperation, depending on the games that are mixed.

1. Introduction

Cooperation is a widespread phenomena in nature [1,2]. Since cooperation involves a benefit to someone and a cost payed by the cooperator, it is hard to explain its establishment by Darwin natural selection. Partial answers can be found within the framework of evolutionary games [3–5]. Usually individuals are players that can choose one of two strategies [5]: cooperation (C) or defection (D). The strategy is adopted during a game round and can be changed in the next round by taking into account the earned payoffs. If the players are set on the vertices of a network, they are constrained to play against their next neighbours with the same strategy or with different strategies [6–10].

In the simplified framework of a two-person game, where only two strategies are available, the outcome of one round of a cooperation game can be represented [11] by the payoff matrix

$$\begin{array}{c|cc} & C & D \\ \hline C & 1 & S \\ D & T & 0 \end{array} \tag{1}$$

If $T>1$ and $S<0$, the game is called prisoner's dilemma (PD). Defection is the best response for both players despite of mutual cooperation being better, what gives rise to a social dilemma where the individual interest is in conflict with the social outcome [12]. When $T>1$ and $S>0$, the game is called snow-drift game (SD) and the best response is to adopt the opposite strategy of the opponent. If $T<1$ and $S<0$, the game is called stag-hunt (SH) and the best response is to adopt the same strategy of the opponent. Note that in the SH game mutual cooperation is the most efficient solution (Pareto efficient) [12]. When $T<1$ and $S>0$, the game is called harmony-game (HG) and mutual cooperation is the best option. The payoff matrix can be represented in a short form by the vector $G=(T,S)$. The parameter space $T\times S$ is shown in the Fig. 1.

Until now much effort has being done to study the case where the same game is played in all interactions [5,13–18]: usually it is always only the prisoner dilemma game, only the snow-drift game, or only the stag-hunt game. But this might not be always the case, as it was already pointed out in some models where the payoff itself was subject to evolution [19–21,24]. The main result in these papers is that, if the payoff is subjected to selection, the population can escape from the dilemma posed by the PD game. Moreover it was shown that Darwinian selection prefers SH to PD game. Here we study a model where in each pairwise interaction two different games can be

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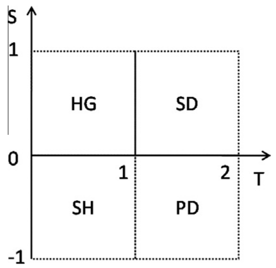

Fig. 1. Payoff parameter space $T \times S$. The harmony game (HG) is represented by the parameters in the range $T < 1$ and $S > 0$, the prisoner's dilemma (PD) in the range $T > 1$ and $S < 0$, the snow-drift (SD) in the range $T > 1$ and $S > 0$, and the stag-hunt (SH) in the range $T < 1$ and $S < 0$.

played in a random fashion, with no selection pressure on the payoffs. The evolution of cooperation in such populations when there is a mixture of different games is more complex than when only a single game is played. Let us approach this new problem in the following way. In the next section we define the evolutionary model. In section III we analyze it in a well-mixed population. We discuss the case of a population structured in a ring (cycle) in section IV. Finally, we present our conclusions in the last section.

2. The model

The evolution is based on two compact processes: the interaction and the imitation process. The population structure defines who interacts with whom. In a well-mixed population each player interacts with a random sample of the population, while in a structured population the players are set on the nodes of a network interacting with their nearest neighbours. We are going to study the mixture of two games, $G_1 = (T_1, S_1)$ and $G_2 = (T_2, S_2)$, just to keep it simple. In each interaction, one round of the game G_1 (G_2) is played with probability w ($1-w$), independently of other rounds. The payoff obtained from each interaction is added to the cumulative payoff of each player.

In the imitation process a player x is randomly chosen to update its strategy. In a well mixed (structured) population the player x randomly selects another player y in the entire population (in the nearest neighbourhood) to compare the cumulative payoffs. Suppose that the cumulative payoffs of the players are Π_x and Π_y. With probability

$$F(\Pi_y - \Pi_x) = \frac{1}{1 + e^{-\beta(\Pi_y - \Pi_x)}},$$

the player x imitates the strategy the player y is using. After the imitation process the payoffs of all of the players are set to zero and the interaction process is started again.

3. Well-mixed population

In an infinite well-mixed population the cooperators have an average payoff given by

$$\Pi_c = x + (1-x)[wS_1 + (1-w)S_2],$$

where x is the fraction of cooperators in the population. The defectors have an average payoff given by

$$\Pi_d = x[wT_1 + (1-w)T_2].$$

So if two different games are played in a well-mixed population, the population dynamics is driven by the average game,

$$G = wG_1 + (1-w)G_2. \tag{2}$$

The mean field equation obtained from the microscopic rules [5] is given by

$$\dot{x} = x(1-x)[F(\Pi_c - \Pi_d) - F(\Pi_d - \Pi_c)]. \tag{3}$$

Although the equilibrium points of the mean field equation are defined by the average game, it is interesting to decouple the dynamics of the two different games. An equilibrium point will be stable or unstable depending on which game has a stronger effect on the dynamics in the region under analysis. In the Fig. 2 it is shown an illustration of the mixture of a SH game and a SD game. Note in the Fig. 2 that if the cooperation fraction is small, the SH game dynamics is stronger and cooperation goes extinct. On the other hand, if the cooperation fraction is close to one, the SD game is stronger and cooperation decreases. The overall effect is that the dynamics of the mixing is equivalent to the PD game dynamics.

In a finite population of size N, a typical quantity of interest is the probability that a single cooperator takes over a population with $N-1$ defectors. This probability is known as fixation probability [4]. The fixation probability of a cooperator in a game $G = (T, S)$ is given by

$$\rho = \frac{1}{1 + \sum_{j=1}^{N-1} e^{(\frac{1}{N} - S)j + \frac{S+T-1}{2N} j(1+j)}}.$$

To calculate the fixation probability when the games $G_1 = (T_1, S_1)$ and $G_2 = (T_2, S_2)$ are mixed, the average payoff of each player is given by the average game $(T, S) = w(T_1, S_1) + (1-w)(T_2, S_2)$. So the fixation

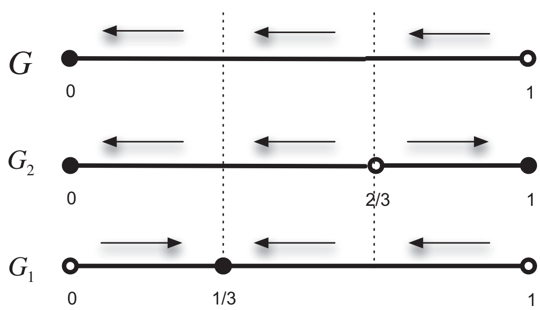

Fig. 2. Dynamics of the fraction x of cooperators for the snow-drift game $G_1 = (T_1, S_1) = (2, 0.5)$, for the stag-hunt game $G_2 = (T_2, S_2) = (0.5, -1)$, and for the mixture $G = wG_1 + (1-w)G_2$ with $w = 0.5$. The average game is a PD game. The stag-hunt game has an effect on the reproduction rate that is stronger than the snow-drift effect when $x < 1/3$ and the cooperation fraction decreases. Both game have the same effect when $1/3 < x < 2/3$ and the cooperation fraction decreases. The snow-drift game has an effect on the reproduction rate that is stronger that the stag-hunt effect when $x > 2/3$ and the cooperation fraction decreases too. The overall result is that cooperation always decreases.

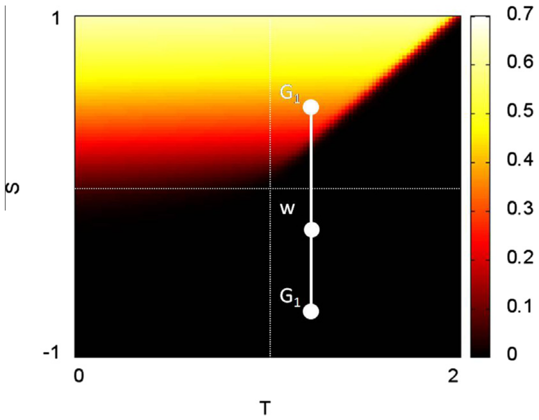

Fig. 3. Fixation probability in the parameter space $T \times S$ for a single game (T, S) and $N = 100$. The fixation probability for the mixture of two games G_1 and G_2 is equivalent to fixation probability of the average game, which is along the line $wG_1 + (1 - w)G_2$ (white continuous line).

probability for the mixture of G_1 and G_2 is given by the corresponding point on the line $w(T_1, S_1) + (1 - w)(T_2, S_2)$, as it is shown in the Fig. 3.

The mixture of games in well-mixed populations can be easily generalized to more than two games, because the overall result can be well approximated by the average game. So if the games $G_1, G_2, \ldots, G_n$ are mixed in a population at probabilities $w_1, w_2, \ldots, w_n$, with $\sum w_i = 1$, the overall result is analogous to the average game represented by the linear combination $w_1G_1 + w_2G_2 + \ldots + w_nG_n$. It is worth noting that in other evolutionary game models the dynamics can be described in an equivalent way by an average payoff matrix as, for example, in the replicator dynamics on graphs in the limit of weak selection [23] and in the evolution of selfishness and other-regarding preferences in lattices [22].

4. Cycle

The cycle is a one-dimensional population structure with periodic boundary conditions. If a single cooperator invades a population of defectors in a cycle, the cooperators grow in a compact cluster, because the strategies that are changed are only those on the two borders of the cooperation cluster. So if n_c is the number of cooperators in the population, the size of the cooperation cluster is also n_c. The evolution of cooperation on the cycle when there is a single compact cluster of cooperators is described by a Markov chain with states $S = \{0, 1, 2, \ldots, N\}$, where the states represent the size of the cooperation cluster.

Let us first consider the dynamics of a single game, with payoffs given by (T, S). If $n_c = 1, 2 \leqslant n_c \leqslant N - 2$, or $n_c = N - 1$, the cooperators are better off only if $2S > T, 1 + S > T$, or $2T > 1 + S$, respectively, as it is shown in the Fig. 4. For instance, if $\beta \to \infty$ a mutant cooperator can invade only if $2S > T$.

In the mixture of the games $G_1 = (T_1, S_1)$ and $G_2 = (T_2, S_2)$, the rate of the transition $k \to k + 1$ is given by

$$W_k^+ = \begin{cases} \frac{1}{N}[w^2F(\Delta_1) + (1-w)^2F(\Delta_4) \\ \quad +w(1-w)(F(\Delta_3) + F(\Delta_2))] & k = 1 \\ \frac{1}{N}[wF(\Delta_5) + (1-w)F(\Delta_6)] & 1 < k < N-1 \\ \frac{1}{N}[w^2F(\Delta_7) + (1-w)^2F(\Delta_{10}) \\ \quad +w(1-w)(F(\Delta_9) + F(\Delta_8))] & k = N-1 \end{cases}$$

and the rate of the transition $k \to k - 1$ is given by

$$W_k^- = \begin{cases} \frac{1}{N}[w^2F(-\Delta_1) + (1-w)^2F(-\Delta_4) \\ \quad +w(1-w)(F(-\Delta_3) + F(-\Delta_2))] & k = 1 \\ \frac{1}{N}[wF(-\Delta_5) + (1-w)F(-\Delta_6)] & 1 < k < N-1 \\ \frac{1}{N}[w^2F(-\Delta_7) + (1-w)^2F(-\Delta_{10}) \\ \quad +w(1-w)(F(-\Delta_9) + F(-\Delta_8))] & k = N-1 \end{cases}$$

The arguments of the function $F(\cdot)$ are shown in the Table 1.

In the limit of large N, the cooperation fraction is given by $x = \frac{i}{N}$. The evolution of the cooperation fraction in the open interval $(\frac{1}{N}, \frac{N-1}{N})$ can be approximated by the mean field equation

$$\dot{x} = x(1 - x)\big(W^+(x) - W^-(x)\big), \tag{4}$$

where $W^+(x)$ and $W^-(x)$ have the same constant values of W_k^+ and W_k^- when $1 < k < N - 1$. Note that the Eq. 4 does not hold when the network has a single cooperator or a single defector. Nevertheless it describes well the dynamics whenever the cooperator cluster has no more than $N - 1$ and no less than one cooperator. The sign of the function $W(x) = W^+(x) - W^-(x)$ determines the dynamics of the equation. If $W(x) > 0$ cooperation increases and if $W(x) < 0$ it decreases.

The sign of $W(x)$ for the parametrized mixture of the games $G_1 = (T, 1)$ and $G_2 = (T, -1)$ is shown in the Fig. 5. Note that with this parametrization, we are looking at two classes of mixture: (i) the harmony game mixed with the SH game and (ii) the SD game mixed with the PD game. If $1 < T < 2$, the SD and the PD games are mixed. If $w < 0.5$ the PD game is played more often and cooperation decreases for all β values. If $w > 0.5$ cooperation increases depending on the T value. The remarkable result is that although increasing T is detrimental for cooperation, the detrimental effect of T is reduced as β gets larger. This result is due to three factors. First, if $w > 0.5$ the SD game is played most of the time. Second, cooperation is the most successful strategy in the SD game given by $G_1 = (T, 1)$. Third, as β gets larger only the most successful strategies are imitated. So, as cooperation is the most successful strategy in the SD game, the higher the selection intensity the more favoured cooperation is. If $0 < T < 1$ the harmony and the SH games are mixed. If $w > 0.5$ the harmony game is played more often and cooperation increases for all β values. On the other hand, if $w < 0.5$ and $0 < T < 1$, the SH game is played more often. Again the temptation T is detrimental for cooperation. But now, in the mixture of $G_1 = (T, 1)$ and $G_2 = (T, -1)$, increasing the selection intensity has an opposite effect: the detrimental effect of T is increased as the β gets larger. This happens because, if $w < 0.5$, the SH game is played more often and cooperations is disfavoured in the SH game. These results are along

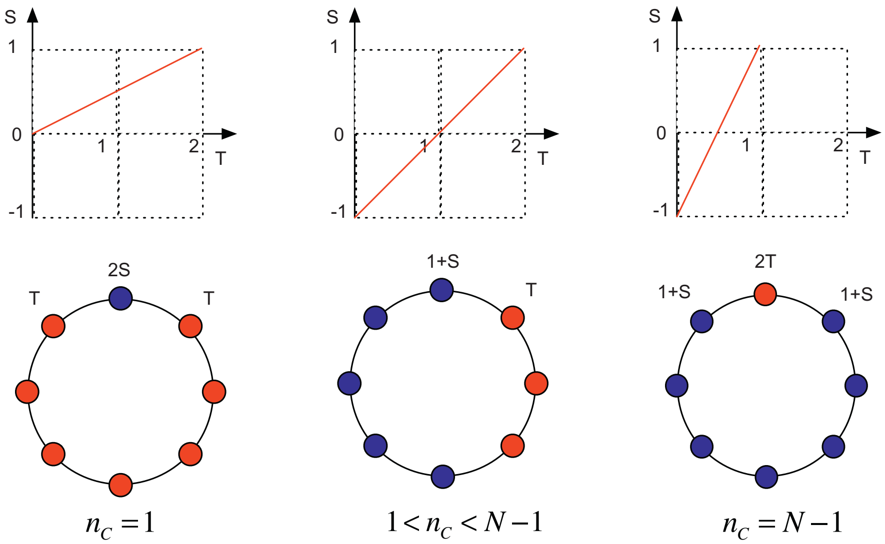

Fig. 4. Schematic representation of the three classes of payoff values on the cycle: $n_c = 1$ (left), $2 \leqslant n_c \leqslant N-2$ (centre) and $n_c = N-1$ (right). The cooperators are in blue and the defectors in red. The (T, S) region where the payoff of a cooperator is larger than the payoff of a defector at the interface between the cooperation and the defection cluster is shown for each class. (For interpretation of the references to colour in this figure caption, the reader is referred to the web version of this article.)

Table 1
Values of arguments of the function $F(\cdot)$ for the transition probabilities W_k^+ and W_k^-

$\Delta_1 = 2S_1 - T_1$	$\Delta_2 = S_1 + S_2 - T_1$
$\Delta_3 = S_1 + S_2 - T_2$	$\Delta_4 = 2S_2 - T_2$
$\Delta_5 = 1 + S_1 - T_1$	$\Delta_6 = 1 + S_2 - T_2$
$\Delta_7 = 1 + S_1 - 2T_1$	$\Delta_8 = 1 + S_1 - T_1 - T_2$
$\Delta_9 = 1 + S_2 - T_1 - T_2$	$\Delta_{10} = 1 + S_2 - 2T_2$

the lines of previous studies showing that the selection intensity β has a significant effect on the evolution of cooperation when a single PD game is played on networks [25] and even when the β itself is subjected to selection [24].

In the limit of small β, also known as weak selection limit, the probability of imitation can be approximated by a linear function depending on the payoff differences. The term $W^+(x) - W^-(x)$ is then approximately given by the difference of the average payoffs of a cooperator and of a defector. The terms can be rewritten in such a way that the average payoff of each strategy is equivalent to the average payoff calculated considering the average game $G = wG_1 + (1-w)G_2$. In the average game $G = (T, S)$, cooperation is favoured as long as $T > 1 + S$, as it is shown in the Fig. 4. So in the weak selection limit the dynamics is the same as the dynamics of the average game, as it is shown in the Fig. 5 for $\beta = 0.1$.

In a finite population the evolutionary process has two absorbing states: full cooperation (state N) and full defection (state 0). The fixation probability of a single cooperator for the parametrized mixture of the games $G_1 = (T, 1)$ and $G_2 = (T, -1)$ can be calculated taken into account the three cases described in the Fig. 4. The analytical result is shown in the Fig. 6. First note that increasing β has a positive effect on the cooperation fixation if $w > 0.5$, though in the parameter region given by $1 < T < 2$ the fixation probability decreases for large β. Nevertheless for large β the population spends most of the time in the cooperative states before reaching the defective absorbing state. This statement can be quantified by looking at the average number of visits to each state before reaching fixation, which can be calculated using the fundamental matrix of

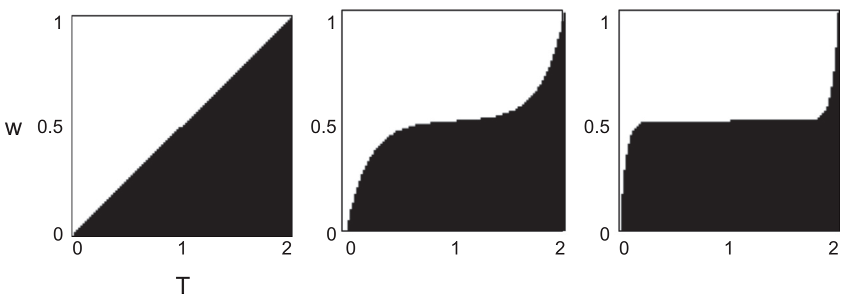

Fig. 5. Signal of the function $W(x)$ in the intervals for the parametrized mixture of the games $G_1 = (T, 1)$ and $G_2 = (T, -1)$ for $\beta = 0.1$ (left), $\beta = 5.0$ (centre), and $\beta = 20$ (right). For each w and T, the sign of $W(x)$ is plotted accordingly to $W(x) > 0$ (white) and $W(x) < 0$ (black).

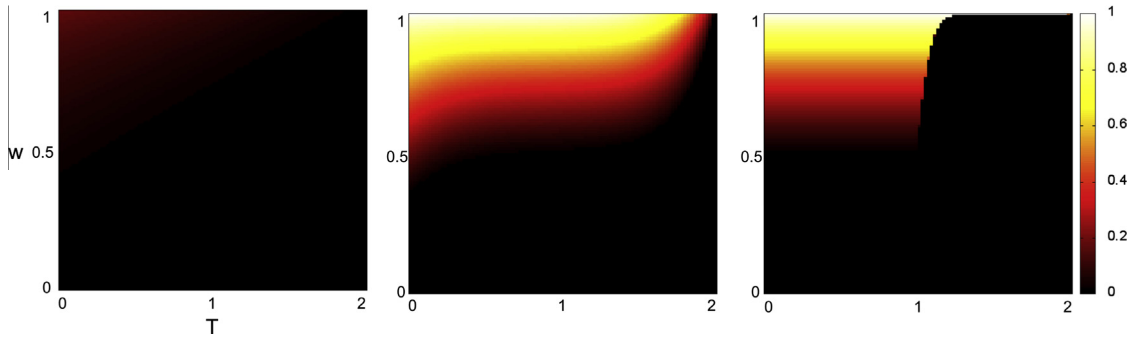

Fig. 6. Fixation probability of a single cooperator for the parametrized mixture of the games $G_1 = (T, 1)$ and $G_2 = (T, -1)$ for $\beta = 0.1$ (left), $\beta = 5.0$ (centre), and $\beta = 100.0$ (right). Here $N = 100$.

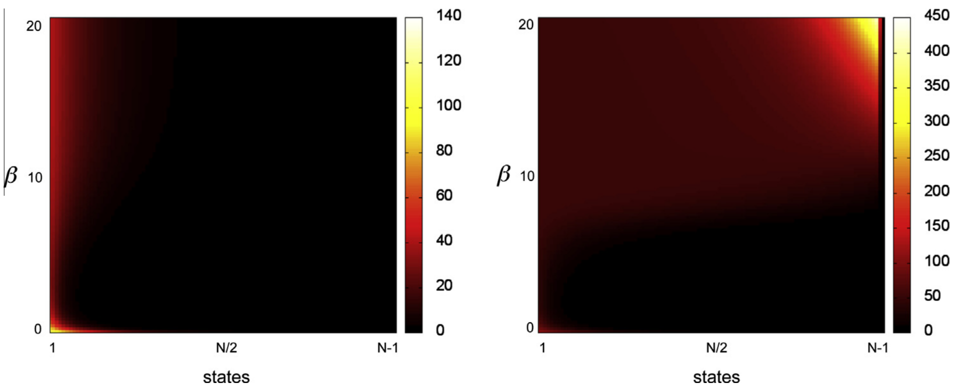

Fig. 7. Number of visits to the transient states for the mixture of $G_1 = (1.2, 1)$ and $G_2 = (1.2, -1)$ with $w = 0.48$ (left) and $w = 0.52$ (right). Here $N = 100$.

the Markov process [26]. The average number of visits to the transient states j when the system starts at the state 1 is shown in the Fig. 7 for the mixture of $G_1 = (1.2, 1)$ and $G_2 = (1.2, -1)$. Note that if $w > 0.5$, the population spends more time in the cooperative states as β gets larger even though it will eventually fixate at the state 0. The basic mechanism behind is that there is a barrier at the state $N - 1$. In the state $N - 1$ the defector out-performs the two cooperating neighbours and, even though cooperation is better than defection in the other states, the barrier at the state $N - 1$ becomes more repellent at large β.

So far we have analyzed the parametrization given by $G_1 = (T, 1)$ and $G_2 = (T, -1)$. The formalism developed here is not restricted to this parametrization. Another interesting parametrization is the mixture of the SH and the PD game, for example, the mixture given by the parametrization $G_1 = (0, S)$ and $G_2 = (2, S)$ in the region $S < 0$. If $w = 1.0$, only the SH game is played. In this parametrization note that cooperation is favoured in the SH game when the population is such that $1 < n_c \leqslant N - 1$. Following the same analytical approach, we can conclude that increasing the selection strength will enlarge the parameter space where cooperation thrives. In other words, increasing the selection strength can promote cooperation for some S values where cooperation would not thrive if the selection was weaker. This parametrization is interesting because it links our work to a previous result where it was shown that, if the SH game is allowed in an evolutionary spatial heterogeneous games [20], cooperation is enhanced.

5. Conclusion

Here we analyzed the evolution of cooperation in mixed games using mean field technics. We showed that in a well-mixed population the mixture of games is equivalent to the average game, with the contribution of each game to the dynamics depending on the cooperation fraction. In a population where more than one game is being played, it might be the case where one game is shaping the imitation process stronger than the others if cooperation is rare, but looses its impact if cooperation is abundant. So the abundance of cooperators is a good indicator of what class of reasoning should be worked out in order to promote cooperation. In the ring topology we showed that, for some mixtures, increasing the selection intensity enlarges the parameter space where cooperation can thrive, while, for other mixtures, it reduces the parameter space. Even though the human social networks are far from being represented by cycles, the analysis of cooperation on the cycle made here is simple, completely analytical and points to the relevant role of networks when more than one game is played.

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